Dihedral Groups

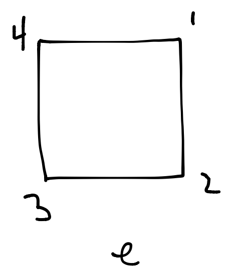
Goal: Study symmetries of an n-gon. (From now on, we write ab in place of $a*b$ for $a,b \in G$, $(a^{-1})^n = a^{-n}$, and $a^0 = e$.)

Example: D_8 (Symmetries of a square)

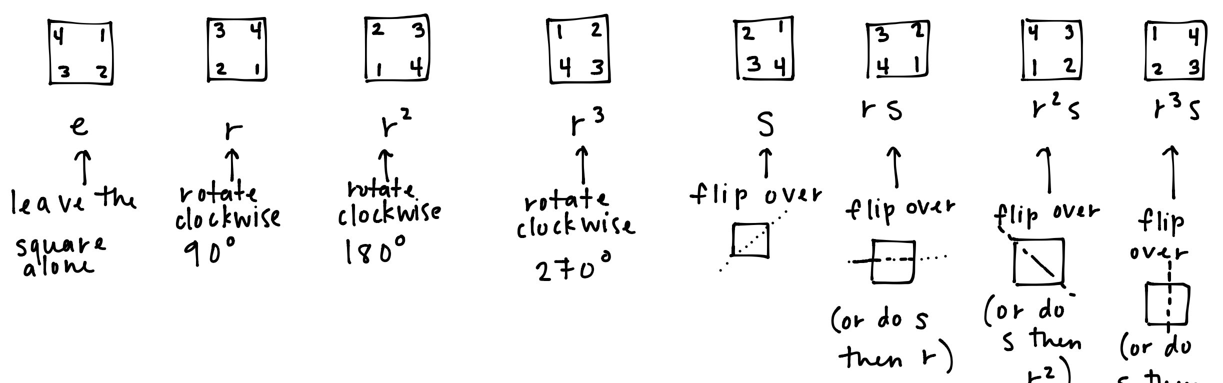
Let D_8 be the set of symmetries of a square.

A symmetry is a rigid motion where the square is replaced so that it exactly covers its original position.

We start w/ the square w/ its vertices labelled:



we can then replace the square in 8 different ways:



Note that the actions are performed from right to left. This is because we are thinking of these as functions on the vertices of the square.

In fact, because our operation is composition of functions, we know its associative. We will soon see why it satisfies the other two axioms.

Here is the multiplication table:

	e	r	r^2	r^3	s	rs	r^2s	r^3s
e	e	r	r^2	r^3	s	rs	r^2s	r^3s
r	r	r^2	r^3	e	rs	r^2s	r^3s	s
r^2	r^2	r^3	e	r	r^2s	r^3s	s	rs
r^3	r^3	e	r	r^2	r^3s	s	rs	r^2s
s	s	r^3s	r^2s	rs	e	r^3	r^2	r
rs	rs	s	r^3s	r^2s	r	e	r^3	r^2
r^2s	r^2s	rs	s	r^3s	r^2	r	e	r^3
r^3s	r^3s	r^2s	rs	s	r^3	r^2	r	e

s row, r column $= sr =$ "do r then s"

1.) Check that e is in fact the identity.

2.) Does every element have an inverse? What is it? i.e. what is $(r^i s^j)^{-1}$?

3.) Is the group abelian?

4.) Notice that every element can be written as $r^i s^j$ for $0 \le i \le 3$, $0 \le j \le i$. What is the "rule" for writing any element this way? i.e. what is $s^m r^n$?

5.) What is the order of each element?

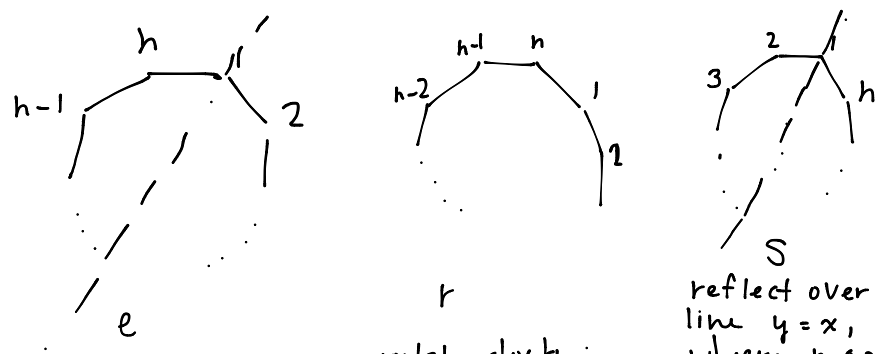
Note that every element can be written in terms of s and r. These are called generators:

Def: Let G be a group. $S \subseteq G$ is a set of <u>generators</u> of G is every element of G can be written as a finite product of elements of S and their inverses. Then G is <u>generated</u> by S.

Dihedral groups in general

D_{2n} is the group of symmetries of an n-gon.

Again, we can generate all the elements by a rotation and a flip.



r: rotate clockwise by $\left(\frac{360}{n}\right)^\circ$

$r(1)=n$, $r(2)=1$, $r(3)=2$, ...

s: reflect over line $y=x$, where n-gon is centered at the origin

$s(1)=1$, $s(2)=n$, $s(3)=n-1$, ...

Properties of D_{2n}

1.) D_{2n} has $2n$ elements.

2.) $e, r, r^2, \dots, r^{n-1}$ are all distinct and $r^n=e$, so $|r|=n$

3.) $s^2=e$.

4.) $s \neq r^i$ for any i.

5.) $r^i s \neq r^j s$ for $i \neq j$ and $i,j \in \{0,1,\dots,n-1\}$

i.e. each element can be written uniquely as $r^m s^h$ for $m \in \{0,\dots,n-1\}$, $h \in \{0,1\}$

6.) $sr = r^{-1}s \; (= r^{n-1}s)$

Thus, s and r don't commute (unless $n=2$), so D_{2n} is not abelian.

7.) $sr^i = r^{-i}s \left(= r^{(n-i)}s\right) \quad \forall \quad 0 \leq i \leq n-1.$

Pf of 4: r^i only fixes 1 if $r^i = e$. s fixes 1, so $s \neq r^i$. $\square$

Pf of 5: s fixes 1, and $r^i(1) = \begin{cases} n+1-i & \text{if } i \neq 0 \\ 1 & \text{if } i = 0 \end{cases}$ Thus, $r^i s(1) = r^j s(1)$
$\iff i = j$. So $r^i s \neq r^j s$ if $i \neq j$. $\square$

Pf of 6: If $sr = r^i$, then $srr^{n-1} = r^{n+i-1} \Rightarrow s = r^{n+i-1}$, which contradicts 4.).

Thus $sr = r^i s$ for some $i \in \{0, \dots, n-1\}$

$r(2) = 1$ and $s(1) = 1$, so $sr(2) = 1$.

Thus, $1 = r^i s(2) = r^i(n) \Rightarrow i = n-1$ $\square$

Pf of 7: We prove $sr^i = r^{-i}s$ by induction on i.

6.) gives the base case

Then $sr^{i+1} \underset{\substack{\uparrow \\ \text{induction} \\ \text{hypothesis}}}{=} (sr^i)r = (r^{-i}s)r = r^{-i}(sr) = r^{-i}(r^{-1}s) = r^{-(i+1)}s.$ $\square$